MAC-CPTM Situations Project

Situation 37: Distributing Exponents

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Prompt

The following scenario took place in a high school Algebra 1 class.   Most of the students were sophomores or juniors repeating the course. During the spring semester, the teacher had them do the following two problems for a warm-up:

1) Are these two expressions, $\left(x^3 y^5\right)^2$ and $x^6 y^{10}$, equivalent? Why or why not?

2) Are these two expressions, $\left(a+b\right)^2$ and $a^2 + b^2$, equivalent? Why or why not?

Roughly a third of the class stated that both pairs of expressions were equivalent because of the Distributive Property.

Commentary

The students appear to be over-generalizing the Distributive Property. They seem to be applying a procedure, "take the number on the outside of the parentheses and multiply it with what is inside of the parentheses," and applying it where the property is not applicable. Therefore the teacher in this situation must have a firm grasp of the Distributive Property (including when it does and does not apply) as well as the rules for exponents. Each of these is addressed in the following Mathematical Foci. Focus 1 contains an explanation of a proper use of the Distributive Property, and Focus 2 examines the exponent rule that applies to this Situation. Focus 3 presents a graphical approach to showing why $\left(x^3 y^5\right)^2$ and $x^6 y^{10}$ are equivalent, and why $\left(a+b\right)^2$ and $a^2 + b^2$ are not. Finally, Focus 4 offers a numerical approach and provides a counterexample to prove that $\left(a+b\right)^2$

and $a^2 + b^2$ are not equivalent.

<u>Mathematical Foci</u>

Mathematical Focus 1: Distributive Property

The Distributive Property states: $a(b + c) = ab + ac$, and $a(b - c) = ab - ac$. This applies to multiplication being distributed over addition (or subtraction). Raising a term to an exponent/power is a different operation from multiplication, and therefore the rules for multiplication (and distribution) cannot be applied in the same way.

Teachers must have a solid understanding of the Distributive Property and in what context it applies. Though exponents at times seem to follow a "distributive rule" (for example, the exponent outside the parentheses in $(x^3y^5)^2$ seems to be "distributed" to get the correct result of x^6y^{10}), teachers must be precise in their mathematical language in order to communicate accurately. Instead of saying that an exponent is "distributed," perhaps "applied" is a better term. That is, the exponent outside the parentheses in $(x^3y^5)^2$ is <u>applied</u> to the x^3 and the y^5.

Mathematical Focus 2: Exponent Rules

The students in this situation do not have a deep understanding of what an exponent is. It may be the case here that the students have an understanding that m^3 means $m \cdot m \cdot m$, but this understanding only goes so far as when the base is a constant or single variable. When the base is more complicated and parentheses are included in the expression, such as $\left(x^3y^5\right)^2$ and $\left(a+b\right)^2$, they may be enticed by the parentheses and the appearance of a number outside of the parentheses to apply what they recalled of a learned procedure: "Take the number on the outside of the parentheses and multiply it with what is inside of the parentheses."

Therefore, teachers and students must know the meaning of exponents and expressions equivalent to $(A)^k$ where A is more complex than a simple constant or a single variable raised to a power.

For problem 1:

$\left(x^3y^5\right)^2$ means "$\left(x^3y^5\right)$ times $\left(x^3y^5\right)$." Using the Commutative Property and the rule for exponents that states $m^a m^b = m^{a+b}$,

$\left(x^3y^5\right)\left(x^3y^5\right) = x^3x^3y^5y^5 = x^6y^{10}$.

Therefore $\left(x^3 y^5\right)^2$ and $x^6 y^{10}$ are equivalent expressions.

For problem 2:

$\left(a+b\right)^2$ means "$(a + b)$ times $(a + b)$." This is a product of two binomials, and the Distributive Property must be used to multiply correctly:

$$\left(a+b\right)\left(a+b\right) = (a+b)(a) + (a+b)(b) = a^2 + ba + ab + b^2 = a^2 + 2ab + b^2$$

Unless a and/or b is equal to zero, $a^2 + 2ab + b^2$ is not equivalent to $a^2 + b^2$.

Mathematical Focus 3: Graphical Approach

In order to look at the graphs of the expressions $\left(x^3 y^5\right)^2$ and $x^6 y^{10}$, it will help to convert the expressions into equations: $\left(x^3 y^5\right)^2 = 1$ and $x^6 y^{10} = 1$, then into functions of *y*:

$$\left(x^3 y^5\right)^2 = 1$$

$$x^3 y^5 = \pm 1 \qquad\qquad\qquad\qquad\qquad x^6 y^{10} = 1$$

$$y^5 = \frac{1}{x^3} \qquad\qquad y^5 = \frac{-1}{x^3} \qquad\qquad y^{10} = \frac{1}{x^6}$$

$$y = \sqrt[5]{\frac{1}{x^3}} \qquad\qquad y = \sqrt[5]{\frac{-1}{x^3}} \qquad\qquad y = \sqrt[10]{\frac{1}{x^6}}$$

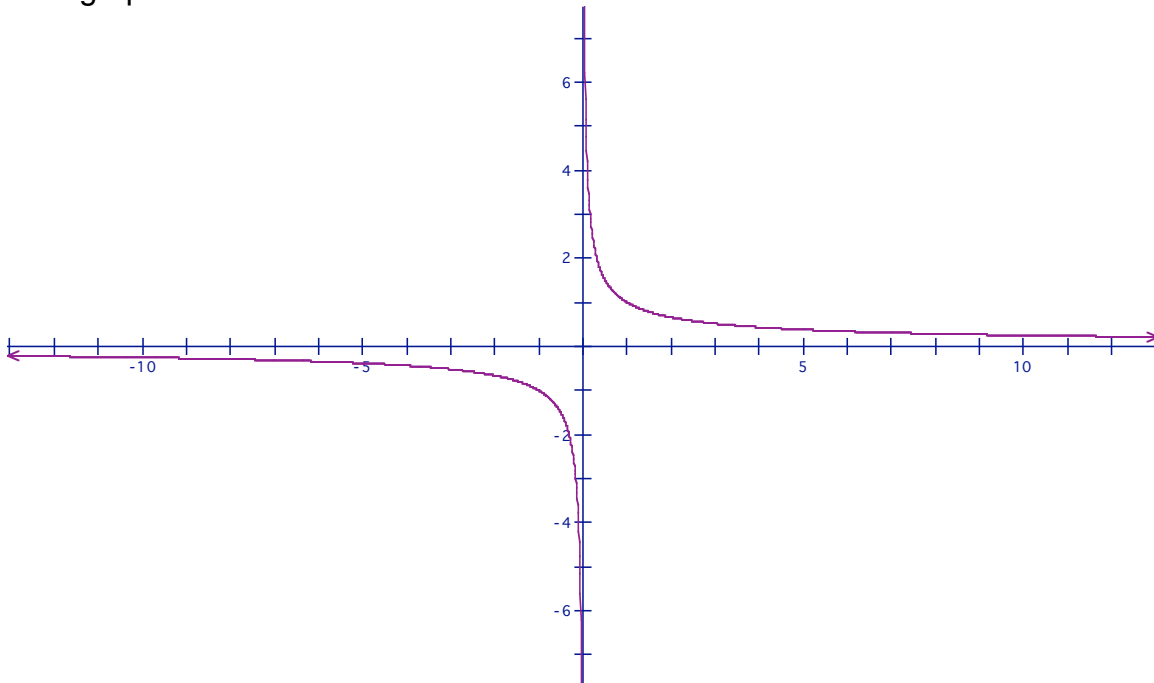Now graph the three functions:
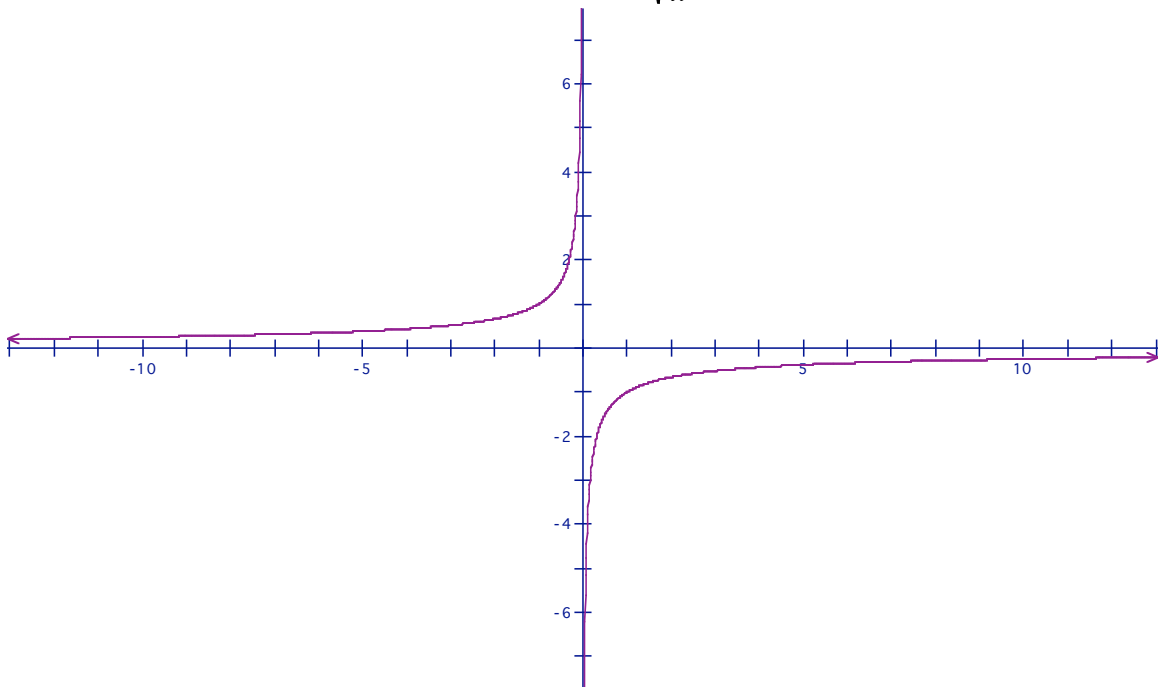
Figure 1: $y = \sqrt[5]{\dfrac{1}{x^3}}$
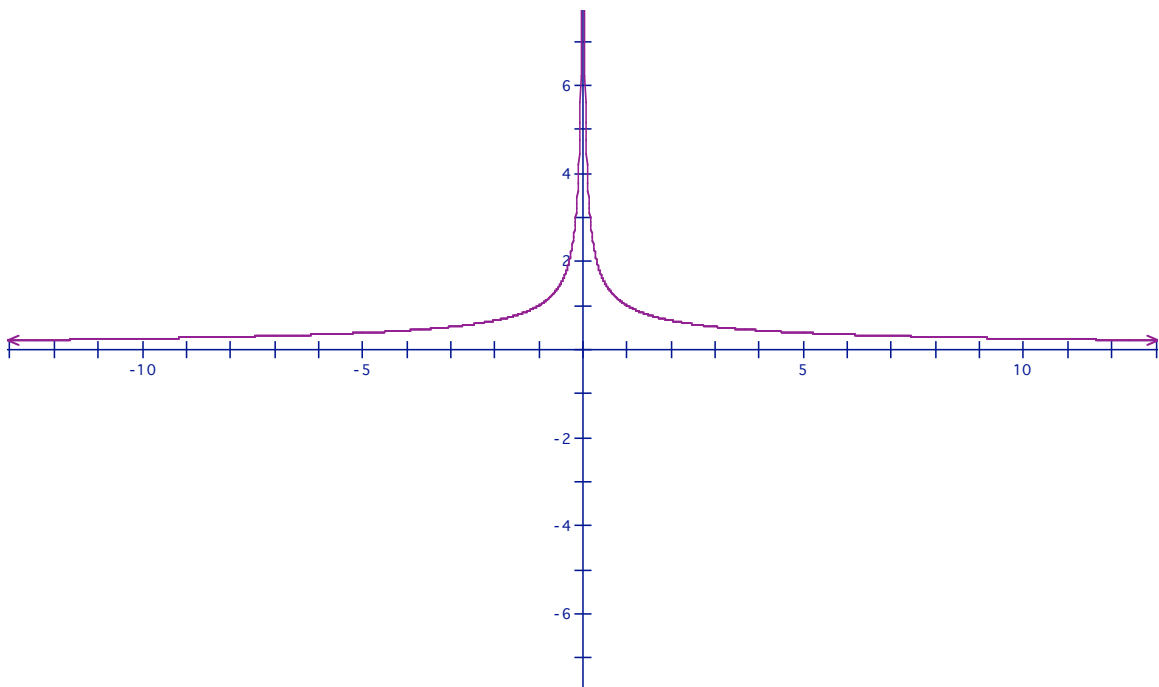

Figure 2: $y = \sqrt[5]{\dfrac{-1}{x^3}}$

Figure 3: $y = \sqrt[10]{\dfrac{1}{x^6}}$

The graphs of $y = \sqrt[10]{\dfrac{1}{x^6}}$ and $y = \sqrt[5]{\dfrac{1}{x^3}}$ are identical when the *x*-values are greater than 0. Therefore, for *x*-values greater than 0, the two expressions are equivalent. The expressions are also equivalent for *x*-values less than 0, as can be seen by comparing the graphs of $y = \sqrt[10]{\dfrac{1}{x^6}}$ and $y = \sqrt[5]{\dfrac{-1}{x^3}}$: they are identical when *x* is less than zero. The range of $y = \sqrt[10]{\dfrac{1}{x^6}}$ is $y > 0$ because the exponent (6) is even.

We will use the same graphical argument for showing that $(a+b)^2$ and $a^2 + b^2$ are not equivalent expressions. First, set both expressions equal to 1 and solve for the variable *b*:

$$(a+b)^2 = 1 \qquad\qquad a^2 + b^2 = 1$$
$$a + b = \pm 1 \qquad\qquad b^2 = 1 - a^2$$
$$b = 1 - a \qquad\qquad b = \pm\sqrt{1 - a^2}$$
$$b = -1 - a$$

Now graph the four functions:

f(x) = 1-x

g(x) = -1-x

h(x) = $\sqrt{1 - x^2}$

q(x) = $-\sqrt{1 - x^2}$
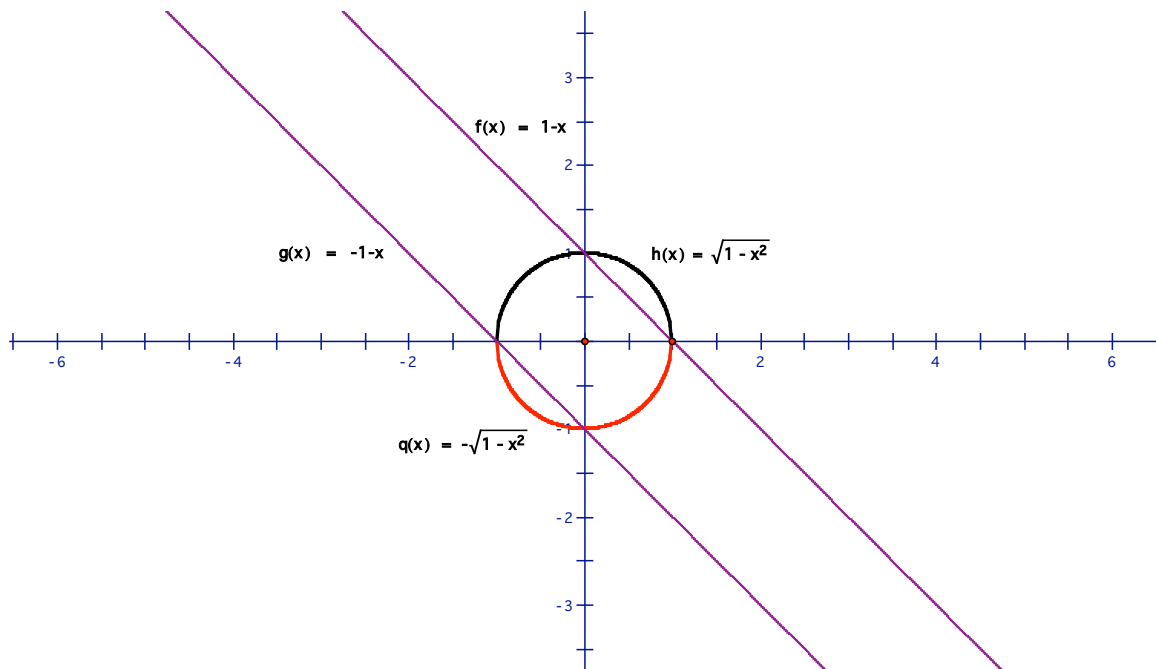
In this graph, the only points of intersection are those points at which either *a* or *b* is equal to 0. Therefore, there are only a limited number of values for which the two expressions are equivalent. Therefore, we cannot say that the two expressions are always equivalent.

Mathematical Focus 4: Numerical Approach

A simple test of a few values is another convincing way to show that $\left(x^3 y^5\right)^2$ and $x^6 y^{10}$ are equivalent, while $(a+b)^2$ and a^2+b^2 are not.

Problem 1:

Let $x = 2$ and $y = 3$.

$$\left(x^3 y^5\right)^2 \qquad\qquad x^6 y^{10}$$
$$= (2^3 \cdot 3^5)^2 \qquad\quad = 2^6 \cdot 3^{10}$$
$$= (8 \cdot 243)^2 \qquad\quad = 64 \cdot 59049$$
$$= (1944)^2 \qquad\qquad = 3779136$$
$$= 3779136$$

Though not a proof, this gives good evidence of the equivalence of the original expressions.

Problem 2:

Let $a = 2$ and $b = 3$

$$(a+b)^2 \qquad\qquad a^2+b^2$$
$$= (2+3)^2 \qquad\quad = 2^2 + 3^2$$
$$= (5)^2 \qquad\qquad = 4+9$$
$$= 25 \qquad\qquad\quad = 13$$

This counterexample <u>does</u> give proof that $(a+b)^2$ and a^2+b^2 are not equivalent expressions.